Game Theory.

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Course: PPE

2. Countries R ("Row") and C ("Column") are at war with each other. Country R's army can strike one (and only one) of three possible targets, cities 1, 2, and 3 in country C. Country C's army can defend one (and only one) of the cities. If country R strikes city m, then city m is destroyed if and only if it is undefended by C. If R strikes city m and it is undefended (and hence destroyed), R receives a payoff of $v_m > 0$ while country C receives $-v_m$. If R strikes city m but does not destroy it (because it is defended by C), both countries earn a payoff of 0. City 1 is most valuable to both countries and city 3 is the least valuable: $v_1 > v_2 > v_3$.

(a) What are the strategies for the players (countries)? What is the payoff matrix of this game? [15%]

Defend

Attack	1	2	3
—	$0, 0$	$V_1, -V_1$	$V_1, -V_1$
=	$V_2, -V_2$	$0, 0$	$V_2, -V_2$
≡	$V_3 -V_3$	$V_3 - V_3$	$0, 0$

Payoff matrix drawn here. Strategy space for both players; choose one city to attack or defend.

$S_{Attack} = \{ -, =, \equiv \}$,

$S_{Defend} = \{ 1, 2, 3 \}$.

Let $(p_1, p_2, p_3; q_1, q_2, q_3)$ be a Nash equilibrium, where $p_i \geq 0$, $i = 1,2,3$, is the probability that R attacks city i (so $\sum_i p_i = 1$) and $q_i \geq 0$ is the probability that C defends city i (so $\sum_i q_i = 1$).

(b) Show that the game does not have a pure-strategy Nash equilibrium. [15%]

This is an anti-coordination game, so it definitely has no pure NE.

Suppose WLOG attacker chose to attack city 1.

Then BR for Column is to choose 1 as well, but that means the attacker will want to choose 2, and so on. No pure mutual BR $\Rightarrow$ no pure NE.

(c) For $i = 1, 2$, prove that if $p_i = 0$ (that is, R does not attack city i), then $p_{i+1} = 0$ (that is, R does not attack city $i + 1$ either). [20%]

Claim: if $p_i = 0$, $p_{i+1} = 0$ for $i = 1, 2$.

Proof:

Let Column be defending with q_1, q_2, q_3.

P1's expected payoff is

Suppose Row were not attacking city 1: $p_1 = 0$, $p_2 \neq 0$. but only 2 and 3. Then the defender's BR is to defend cities 2 and 3 only. But if that's the case, then Row would prefer to attack city 1! The maximum value the attacker can get mixing btw 2 and 3 is $\bar{v}_2$ at the most. But a deviation gives v_1 which is profitable.

Similarly, if $p_2 = 0$ and $p_3 \neq 0$, then the defender is attacking city 1 and city 3. But here it would always be better to attack city 1 and city 2;

in a MSNE you have some probability of looting city 1, or city 2/3; $v_2 > v_3$. hence profitable deviation.

and his opponent's payoff is

$$U(C) = p_2(q_1+q_3) - v_2 \quad + p_3(q_1+q_2) - v_3 .$$

But if so, then Column has a deviation to set $q_1 = 0$. But if so, then p_1 must be 1. This is a contradiction.

(d) Use the previous result to show that there is no mixed Nash equilibrium in which exactly cities 1 and 3 are attacked with positive probabilities; that is, p1 , p3 > 0 with p1 + p3 = 1 is impossible. Also show that there is no mixed Nash equilibrium in which exactly cities 2 and 3 are attacked with positive probabilities (p2, p3 > 0 with p2 + p3 = 1 is impossible as well).
[15%]

Since if $p_2 = 0$, $p_3 = 0$, there can be no MSNE where $p_1 > 0$ and $p_3 > 0$. because $p_2 = 0$ and $p_3 \neq 0$ which is a contradiction.

Similarly, if $p_1 = 0$, $p_2 = 0$, hence an MSNE with $p_1 = 0$, $\underline{p_2 > 0}$, $p_3 > 0$ is a contradiction.

Shown. $\square$

The results established so far imply that in any Nash equilibrium, either all three cities are attacked with positive probabilities or only cities 1 and 2 are attacked with positive probabilities.
(e) Characterize the conditions under which there is a Nash equilibrium in which only cities 1 and 2 are attacked with positive probabilities (i.e., p1 , p2 > 0, p1 + p 2 = 1).
[15%]

Both players must be playing their mutual BRs. This implies that if only cities 1 and 2 are being attacked, only those cities can be defended because otherwise the defender has a profitable deviation to never defend city 3 which is never attacked.

For this to be an NE it must be that the expected payoff from this mixed strategy for the attacker is weakly greater than v_3, i.e attacking city 3 for sure. Hence,

$$p_1(1-q_1)v_1 + (1-p_1)v_2 \geq v_3 .$$

This means that v_1 and v_2 have to be large enough compared to v_3.

(f)Characterize the conditions under which there is a Nash equilibrium in which all cities are attacked (i.e., p 1, p2 , p3 > 0).
[20%]

The opposite of the condition must hold:

$$p_1(1-q_1)v_1 + (1-p_1)v_2 < v_3 .$$

Thus the attacker has a profitable deviation to attack City 3 only, hence a mixed eqm attacking only two cities (1 and 2) can't be sustained.

4. Two firms produce the same product whose price in period $t = 0, 1, \ldots, 9$ is $P_t(Q_t) = 10 - t - Q_t$ or 0, whichever is greater, where Q_t is the total output in period t. Note that the price can never be negative.

In each period, firm i either produces a quantity equal to its fixed capacity or nothing. Once a firm exits, producing 0 at a given t, it cannot re-enter: it has to produce 0 at all $t' > t$. The firms have identical unit costs, $c = 0.99$ per unit produced, but different capacities: $k_1 = 4$ and $k_2 = 2$, respectively. For instance, if both firms operate at $t = 0$ then the price is $10 - 6 = 4$; the firms' revenues are 16 and 8, whereas their costs are $4c$ and $2c$, respectively. The values of c, k_1 and k_2 are commonly known. A firm's profit in period t is simply the difference between its revenue and cost in period t.

In each period, the firms decide individually and simultaneously whether to produce (at capacity) or exit for good; past actions are observable to both firms. Each firm's continuation payoff at t, which it maximizes at t, is the undiscounted sum of its profits at and after t. There are no fixed costs; a firm that exits at t makes zero profit at t and in every period thereafter.

(a) Describe precisely but succinctly the firms' possible strategies. (You are not required to list all strategies.) [15%]

At time t, a firm observes whether or not the other firm has exited, and either enters, exits, or mixes. So the strategy is state contingent and specifies an action at each time step t whether In or Out depending on whether the other firm exited. So, it looks like this:

$$\left\{ \begin{array}{l} t=0: \{\text{In}/\text{Out}/\text{Mix}\} \\ t=1: \left\{ \begin{array}{l} \text{if opp exited } \{\text{In}/\text{Out}/\text{Mix}\} \\ \qquad\quad \text{didn't } \{\text{In}/\text{Out}/\text{Mix}\} \end{array} \right\} \\ \vdots \\ t=9: \left\{ \begin{array}{l} \text{if opp exited } \{\text{In}/\text{Out}/\text{Mix}\} \\ \qquad\quad \text{didn't } \{\text{In}/\text{Out}/\text{Mix}\} \end{array} \right\} \end{array} \right\}$$

(b) Let t_i denote the last period in which firm i makes a strictly positive profit by producing its capacity provided the other firm is no longer operating. Determine the values of t_1 and t_2 and explain what firm i's strategy specifies in any subgame-perfect equilibrium (SPE) at every $t > t_i$. [15%]

$$P_i(Q_t) = 10 - t - Q_t$$

$$Pq_i - Cq_i$$

$$\Pi_i(Q_t) = [P_i(Q_t) - c] \cdot q_i$$

$$= (10 - t - Q_t - 0.99) \cdot q_i$$

For firm 1,

$$= (10 - t - 4 - 0.99) \cdot 4$$

Firm only stays in when $\Pi_i > 0$, hence here

$$10 - t - 4 - 0.99 > 0,$$

$$t < 5.01 \text{ (approx)}.$$

And for firm 2,

$$10 - t - 2 - 0.99 > 0,$$

$$t < 7.01 \text{ (approx)}$$

Hence any SPE must specify {out} at $t > 5$ and $t > 7$ for firms 1 and 2 respectively.

(c) Explain what firm 2's strategy specifies in any SPE in periods $t = t_1 + 1, \ldots$. Explain whether firm 2's strategy in any SPE at $t \geq t_1 + 1$ is contingent on firm 1's behaviour observed before t. [15%]

Firm 2 will play {In} in periods 6 and 7. because it makes positive profit, since Firm 1 has exited.

Firm 2's strategy in any SPE after $t \geq t_1 + 1$ doesn't depend on firm 1's behaviour because in any SPE, firm 1 always plays {Out} at $t = 6$. (Any threat from firm 1 to stay is not credible — firm 2 should disregard it.).

(d) What is firm 2's best response at $t = t_1$ to firm 1's action at t_1? Find out what each firm's SPE strategy specifies at $t = t_1$. [15%]

If firm 1 plays {In}, firm 2 will play {Out}.

In time $t = 5$ the stage game looks as follows:

	In	Out
In	-7.96, -1.94	0.04, 0
Out	0, 6.06	0, 0

If firm 1 plays $\{In\}$, then

$$\begin{aligned}\Pi_2(Q_5) &= \left[P_i(Q_2) - C(q_2)\right] \cdot q_2 \\ &= \left[10 - t - (q_1 + q_2) - 0.99\right] \cdot q_2 \\ &= (10 - 5 - 6 - 0.99) \cdot 2 \\ &= -3.98\end{aligned}$$

But in period 6 and 7 it makes

$$(10-6-2-0.99)2 + (10-7-2-0.99)2$$

and hence its total payoff is $= -1.94$.

In this stage both firms mix to keep both firms indifferent. Here firm 1 is indifferent when firm 2 plays In with probability p:

$$p(-7.96) + (1-p)0.04 = 0$$

$$p = 1/200$$

And firm 2 is indiff when firm 1 $\{In\}$ with prob q:

$$q(-1.94) + (1-q)6.06 = 0$$

$$q = \frac{303}{400} \approx 3/4.$$

Hence each firm's SPE strategy asks them to mix, $\{1/200, 199/200\}$, $\{303/400, 97/400\}$ respectively.

(e) Complete the analysis of the unique SPE of the game for all $t < t_1$. Describe the SPE strategies precisely. [20%]

In periods 0, 1, 2, 3, both firms play {In}.

In period 4, both firms mix.

5, both firms mix, $\{1/200, 199/200\}$, $\{303/400, 97/400\}$, or the firm still left plays {In} if the other firm exited.

6 and 7, firm 2 plays {In}
firm 1 {Out}.

8, 9, firm 2 exits.

(f) Outline how the analysis in parts (b)–(d) changes if the firms' unit cost is c0 = 1.99. Do not complete the derivation of all subgame-perfect equilibria in this case.
[20%]

The only thing that changes is that in period 5, firm 1 plays {Out}, and in period 4, the firms mix with $\{1/200, 199/200\}$, $\{303/400, 97/400\}$

6. (a) Explain carefully what is meant by an Evolutionarily Stable Strategy in a symmetric two-player game. [10%]

In a symmetric two-player game, a strategy α^* is an ESS if the following two conditions hold:

1. (α^*, α^*) is a symmetric Nash equilibrium of the game;

2. If there exists another strategy $\beta \neq \alpha^*$ that is a BR to α^*

$$(u(\beta, \alpha^*) = u(\alpha^*, \alpha^*),$$

then it must be worse against itself:

$$u(\beta, \beta) < u(\beta, \alpha^*) = u(\alpha^*, \alpha^*).$$

(b) Determine which if any strategies are evolutionarily stable in the following game:

	L	R
L	a, a	$0, c$
R	$c, 0$	b, b

where the row player's payoff is listed first, $0 < b < a$ and $0 < c < a$. [20%]

$a > b > 0$

$a > c > 0$.

		p	$1-p$
		L	R
p	L	$\underline{a}, \underline{a}$	$0, c$
$1-p$	R	$c, 0$	$\underline{b}, \underline{b}$

Let's find all NEs and see which are ESS.

L is an ESS b/c $\{L, L\}$ is a pure strict NE.

R is also an ESS b/c $\{R, R\}$ is also a pure strict NE: deviating nets 0.

There is also a mixed equilibrium: Each player is indifferent when his opponent plays L with probability p, when

$$\Rightarrow \quad \overbrace{p(a) + \cancel{(1-p)}\,0}^{L} = \overbrace{p(c) + (1-p)\,b}^{R}.$$

$$\Rightarrow \quad pa = pc + b - pb$$

$$\Rightarrow \quad pa + pb - pc = b$$

$$\Rightarrow \quad p(a + b - c) = b$$

$$\Rightarrow \quad p = \frac{b}{a+b-c}.$$

This gets a payoff of $\frac{b}{a+b-c}\,a$, which is <u>not</u> an ESS because a mutant playing L gets a, with itself, but $\frac{b}{a+b-c}\,a < a$ by the inequality that $a > c$.

(c) Explain what is meant by the replicator dynamic in a symmetric game played by a single population and discuss when it might predict the evolution of play well. Write down the replicator dynamic for the game in the previous part and discuss how play evolves under it. [20%]

The replicator dynamic describes how the proportion of players playing a strategy changes over time depending on how many other players are playing it.

It might predict the evolution of play well when agents indeed follow such a best response and update strategy.

Let p be the proportion of players playing L.
The payoff to playing L against the population strategy is ap, while the population strategy earns $ap^2 + \cancel{0(p)(1-p)} + c(1-p)(p) + b(1-p)^2$

$$= ap^2 + c(1-p)(p) + b(1-p)(1-p)$$
$$= ap^2 + (1-p)(cp + b - bp)$$
$$= ap^2 + (1-p)[(c-b)p + b]$$

against itself, so

$$\dot{p} = p\left(ap - ap^2 - (1-p)(c-b)p + b)\right)$$
$$= p\left(ap(1-p) - (1-p)[(c-b)p + b]\right)$$
$$= p(1-p)(ap - (c-b)p - b)$$
$$= p(1-p)(ap - cp + bp - b)$$
$$= p(1-p)(p(a+b-c) - b)$$

$\Rightarrow p = 0$, $p = 1$, and $p = \frac{b}{a+b-c}$

are rest points. If

$p > \frac{b}{a+b-c}$ initially, we converge

to $p = 1$ (because $\dot{p} > 0$), otherwise to $p = 0$.

(d) Now suppose that the game in part (b) is played by a finite population of fixed size. Each player in the population plays either L or R. Each period one of the players has the opportunity to revise their strategy, with all players equally likely to be chosen for this opportunity. With probability $1 - \varepsilon$ the chosen player chooses a best reply to the distribution of strategies in the population, with probability ε she chooses each strategy with probability $1/2$. If ε is small but positive, explain carefully how you would expect the system to evolve in the long run.

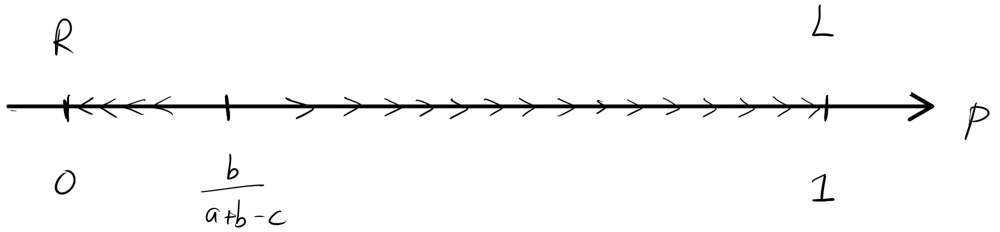
Discuss whether these results help predict play in this game in an economic context. [50%]

The long-run behaviour of the system is independent of the initial state. This is because in the long run the system will reach one of the rest points because ε is small.

Note that the rest point $p = \frac{b}{a+b-c}$ is unstable, because any perturbation from it will cause the system to move towards either one of the rest states $p = 0$ or $p = 1$.

Because ε is small, most of the time will be spent in either $p = 0$ or $p = 1$ equilibria, as you need on the order of $\frac{b}{a+b-c}N$ mistakes to go to the other equilibrium.

Which equilibrium is more attractive depends on the relative values of a, b, and c. If a is relatively large such that $\frac{b}{a+b-c}$ becomes small, then most of the time will be spent at L, and vice-versa. See diagram.



Does this predict play in an economic context?

It seems strange to only allow one agent to change their strategy at every period, which seems more suited for biological adaptation (random mutations take time and usually don't happen very often).

The idea of biological mutation and fitness functions may be inappropriate in an economic context.

8. Alice and Bob play the following stage game; the analysis will involve its repetition over several periods.
Each player starts the stage game with a budget of 2 payoff units. Simultaneously, each may either "enable" a donation to the other (action E) or not (action N). Enabling a donation to the other reduces the player's own payoff by 1 and increases the other's payoff by 2; this is in addition to any change that the other's action may induce in their payoffs. Action N does not affect payoffs. Alice (but not Bob) has a third option: instead of E and N she may play action D ("destroy"), which makes both players lose everything they could have earned in the stage game – that is, the stage-game payoff of each player is reset to zero.
(a) Write down the normal form of the stage game (hint: it is simply a 3x2 table) and find all of its Nash equilibria.
[10%]

a)

		p E	$1-p$ N
p	E	3 , 3	1 , $\underline{4}$
$1-p$	N	$\underline{4}$, 1	$\underline{2}$, $\underline{2}$
	D	0 , 0	0 , 0

The only pure NE is $\{N, N\}$.

MSNE : Observe that D is strictly dominated for Alice and hence we don't have to consider it in NE.

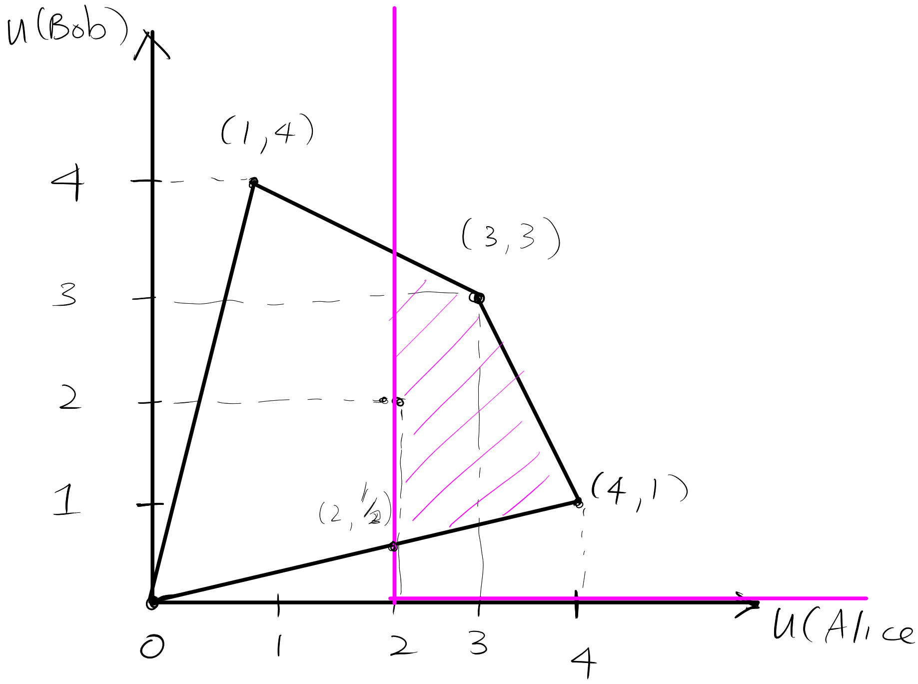
This is the Prisoner's Dilemma : no MSNE, N strictly dominant.

(b) Derive (not just assert) each player's minmax payoff and plot the set of feasible and individually rational payoffs of the stage game.
[15%]

Bob's minmax payoff is trivially 0; once Alice plays D, Bob can get $\bar{0}$ at most no matter what he plays.

Alice's minmax payoff is 2. Notice that N is strictly dominant so she gets $\underline{2}$ always by playing N, and Bob can make sure she gets $\bar{2}$ at most by always playing N.

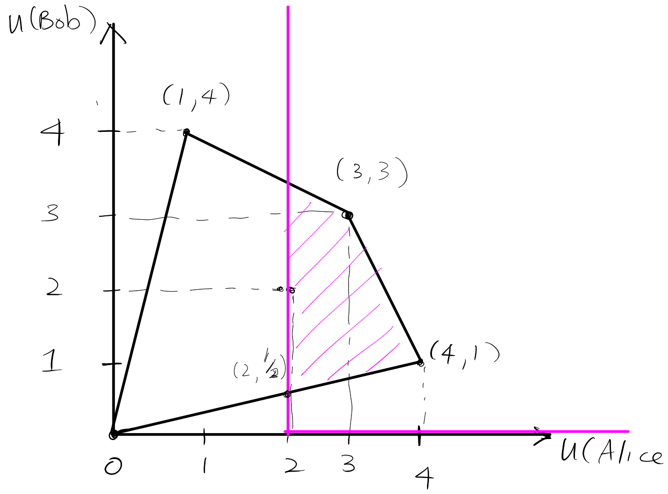
Set of individually rational payoffs:



	E	N
E	3, 3	1, 4
N	4, 1	2, 2
D	0, 0	0, 0

The set of feasible payoff is the convex hull and the individually rational payoffs are the pink shaded area where payoffs exceed the minmax value.

In what follows, assume that the stage game is repeated infinitely many times. Players observe all past actions. Each player maximizes the discounted sum of his or her stage-game payoffs using discount factor d 2 (0, 1).
(c) What are the lowest and highest sums of the players’ per-period payoffs that can be sustained in a subgame-perfect equilibrium (SPE) of the infinitely-repeated game for d close to 1? Explain.
[15%]



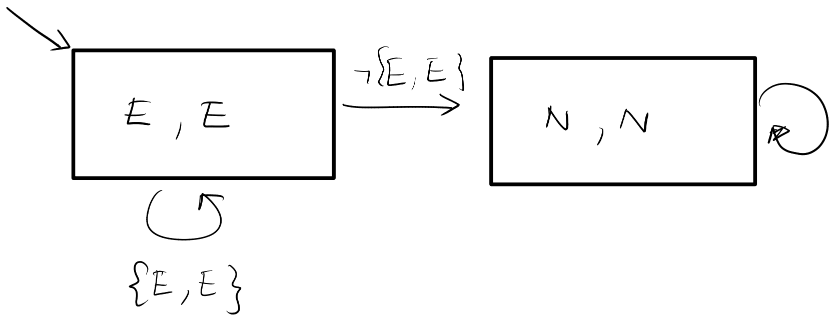
Theorem (D. Fudenberg and E. Maskin, 1986):
Assume V^* (set of feasible and strictly IR payoffs) is n-dimensional. For any $(v_1, \ldots, v_n) \in V^*$ there is $\underline{\delta} \in (0,1)$ such that for all $\delta \geq \underline{\delta}$ the infinitely repeated game has a subgame perfect equilibrium with average discounted payoffs $(v_1, \ldots, v_n)$.

Fudenberg and Maskin (1986)'s Perfect Folk Theorem: any payoff in the set of feasible and IR payoffs is feasible for $\delta \to 1$.

Hence, the highest sum is $(3+3)=6$, and the lowest sum is $(2+1/2 = 5/2)$. The intuition being that any payoff that is not IR can't be SPE, because you could get a higher payoff in that subgame by playing to obtain your minmax value. Additionally, it's not credible for Alice here to minmax Bob forever as she will want to deviate.

(d) Construct a SPE for d sufficiently close to 1 such that both players choose E in every period on the equilibrium path. What is the lowest d for which there exists a SPE (not just the one you constructed) that induces (E, E) in every period? Explain.
[20%]

Simple Nash reversion will do it for $\delta \to 1$.



No profitable oneshot dev at $\boxed{E, E}$ iff

$$4(1-\delta) + 2(\delta) < 3$$

$$\Rightarrow 4 - 4\delta + 2\delta < 3$$

$$\Rightarrow \delta > 1/2,$$

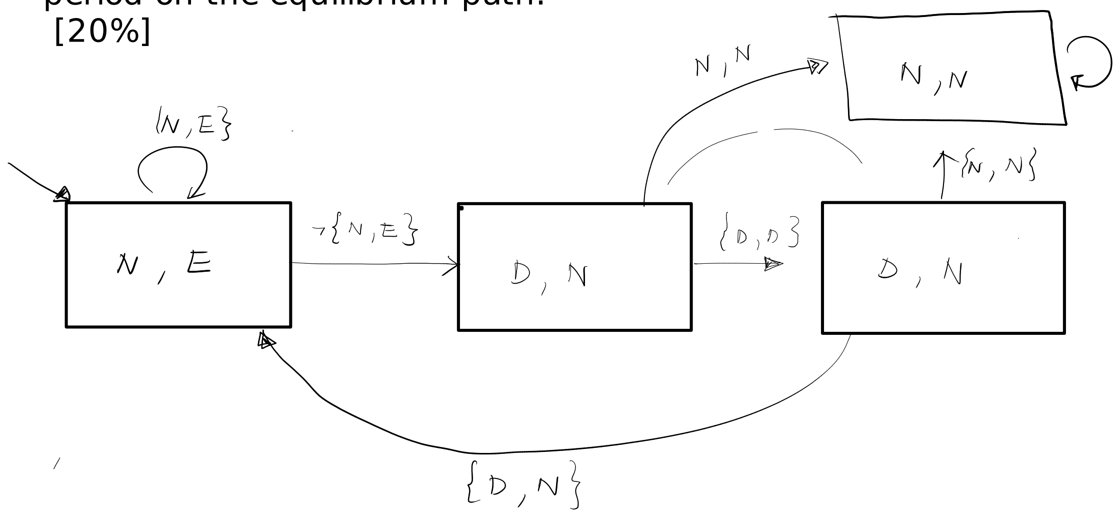
and you never deviate at $\boxed{N, N}$ as it's NE.

What is the lowest d for which there
exists a SPE (not just the one you constructed) that induces (E, E) in every
period? Explain.

The harshest punishment possible for Alice must give her a continuation payoff of 2, because Alice guarantees her minmax payoff of 2 otherwise. And hence the SPE I've constructed, which has ADV = 2, gives the lowest $\delta = 1/2$.

Note Bob's lowest continuation ADV is $1/2$, even lower than Alice's, so the δ needed for Bob not to deviate is even lower.

(e) Construct a SPE for d = 0.75 in which Alice plays N and Bob plays E in every period on the equilibrium path.
[20%]



	E	V
E	3 , 3	1 , 4
N	4 , 1	2 , 2
D	0 , 0	0 , 0

To verify this SPE, we check every state of this FSM and see if anyone has a profitable one-shot deviation

At $\boxed{\{N,E\}}$, Bob doesn't deviate iff

$$2 + 0\delta + 0\delta^2 + \delta^3\left(\frac{1}{1-\delta}\right) < \frac{1}{1-\delta}$$

When $\delta = 3/4$,

$$2 + (3/4)^3(4) < 4$$

$$\Rightarrow (3/4)^3 < 1/2$$

which is true. Hence no incentive to deviate.

Check the punishment phase: at $\boxed{D, D}$, Bob never deviates. Alice doesn't deviate if

$$2/_{1-\delta} < 0 + 0 + \delta^2\left(\frac{4}{1-\delta}\right)$$

$$\Rightarrow 2 < 4\left(\frac{9}{16}\right)$$

which is true.

Hence this is an SPE.

(f) What is the lowest d such that (N, E) in every period (on the equilibrium path) can be sustained in some SPE of the infinitely-repeated game? Explain whether this is different from the lowest d found in part (d), and why. [20%]

The lowest δ is when Bob doesn't want to deviate, and that is when

$$2 + 0 + \dots + \delta^N \left(\frac{1}{1-\delta}\right) < \frac{1}{(1-\delta)}$$

$$2(1-\delta) + \delta^N < 1$$

$$1 < 2\delta - \delta^N$$

The lowest ADV that can be sustained depends on Alice's willingness to punish. Alice can punish for N periods as long as

$$2/_{1-\delta} < 0 + 0 + 0 + \dots + \delta^N \left(\frac{4}{1-\delta}\right)$$ holds,

in other words

$$\delta^N > 1/2,$$

$1 < 2\delta - \delta^N$ and $\delta^N > 1/2$.

Solve for $\min(\delta)$ such that this is true.

This δ will be higher, because Bob's expected payoff is lower in $\boxed{N, E}$ — he will want to deviate more.